Bremsstrahlung Effects around Evaporating Black Holes

[1]Don N. Page*, [2,3]B. J. Carr†, and [4]Jane H. MacGibbon‡

[1]*Department of Physics, University of Alberta, Edmonton, Alberta T6G 2G7, Canada*
[2]*Astronomy Unit, Queen Mary, University of London, Mile End Road, London E1 4NS, UK*
[3]*Research Center for the Early Universe, Graduate School of Science, University of Tokyo, Tokyo 113-0033, Japan*
[4]*Department of Chemistry and Physics, University of North Florida, Jacksonville, Florida 32224, USA*
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We discuss a variety of bremsstrahlung processes associated with charged particles emitted by evaporating black holes. We show that such particles produce a negligible number of bremsstrahlung photons from their scattering off each other, though at low frequencies inner bremsstrahlung photons dominate over the direct Hawking emission of photons. This analysis and the further analysis of the accompanying paper invalidate Heckler's claim that sufficiently hot evaporating black holes form QED photospheres and have similar implications for putative QCD photospheres.

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I. INTRODUCTION

Hawking [1, 2] has shown that black holes emit thermal radiation, which for small enough black holes can include massive particles like electrons and positrons. Heckler [3, 4] has argued that bremsstrahlung processes involving such emitted charged particles will produce a large number of photons for each initial charged particle and lead to a quasithermal photosphere. Here we show that the two-body bremsstrahlung (one charged particle scattering off the electromagnetic field or virtual photons of another charged particle and emitting one or more free photons) is suppressed by small factors besides the α^3 factor from the minimum number of three photon vertices and is not enhanced by any large factors such as the gamma factors of the relativistic charged particles, when the black hole temperature is large compared with the charged particle rest masses. Therefore, the fraction of the initial charged particle energy that will go into two-body bremsstrahlung photons is expected to be of the order of 10^{-8} or less.

One of these suppression factors is the ratio of the reduced Compton wavelength of the emitted particles to their average radial separation. This ratio is about 5.7×10^{-3} for relativistic electrons and positrons emitted by a small hot black hole whose mass is much less than 10^{17} grams. Another factor that may be small for relativistic charged particles (and which goes inversely with their gamma factor E/m) is the causality suppression: The electromagnetic field of one charged particle emitted by the black hole cannot scatter another charged particle, leading to the emission of a bremsstrahlung photon, until there has been time for the electromagnetic field to propagate causally from the first particle to the second. This greatly reduces the classical estimate of the expected momentum transfer between the two particles from what it would have been if the particles had always existed and had come in from infinity with the same energies and impact parameter. Thus two-body bremsstrahlung is suppressed by two small factors (and not enhanced by any large factors) besides the α^3 suppression. Therefore, the two-body bremsstrahlung certainly does not seem to be anywhere nearly sufficient to lead to a photosphere.

There is another bremsstrahlung process that is also small but greater than the two-body bremsstrahlung process that was incorrectly conjectured to lead to a photosphere. This is inner bremsstrahlung, produced by the emitted charged particles as they change their velocities from initially being effectively at rest in the frame of the black hole (or a distant observer) to asymptotically moving radially outward at nearly the speed of light. This process has only the single photon vertex of the emitted inner bremsstrahlung photon and so is suppressed by only one power of α. However, it is enhanced by a logarithm of the gamma factor of the emitted charged particle. Therefore, the inner bremsstrahlung emitted by relativistic electrons and positrons can give photon luminosities that are tens of percent as large as the direct Hawking photon emission. If a black hole is hot enough to emit many massive charged particle species, then the inner bremsstrahlung emission can have even more total power than the direct photon emission.

Since the direct photon emission is suppressed at wavelengths long compared with the size of the hole (giving a very

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blue spectrum at low frequencies), whereas the inner bremsstrahlung spectrum is essentially a white spectrum up to a cutoff near the black hole temperature, the inner bremsstrahlung always dominates at sufficiently low frequencies. For example, we shall show below that a black hole of mass 5.0×10^{14} grams, whose lifetime equals the current age of the Universe, has a direct photon total power of about 2.6 gigawatts, but only about 2.4×10^{-29} watts is in the visible range because of the low-frequency suppression. On the other hand, the inner bremsstrahlung photons from such a hole would give about 4.0 watts of power in the visible range, much greater than that of the direct photons in this low-frequency band.

Because the inner bremsstrahlung always greatly dominates over the two-body bremsstrahlung (which has a relative suppression factor at least as small as α^2 divided by a logarithm of the gamma factor), we may regard inner bremsstrahlung as the basic bremsstrahlung process and the other bremsstrahlung processes as merely giving small corrections to it. For example, the two-body bremsstrahlung may be considered as two-body scattering of the outgoing charged particles that changes their asymptotic momenta, and hence the resulting spectrum of inner bremsstrahlung photons, only very slightly. In total, however, the inner bremsstrahlung always just adds a fraction of order unity to the direct photon power and does not produce a large number of photons of energy comparable to that of the charged particles. Therefore, charged particle bremsstrahlung cannot lead to a photosphere as Heckler conjectured, as we shall see by the more detailed arguments below and as is discussed in our accompanying paper [5].

II. TWO-BODY BREMSSTRAHLUNG IN TERMS OF IMPACT PARAMETERS

In this paper we shall consider bremsstrahlung by electrons and positrons emitted by a Schwarzschild black hole of temperature $T_{bh} = (8\pi M_{bh})^{-1}$ much greater than the electron rest mass m_e. We use Planck units with $\hbar = c = G = k = 4\pi\epsilon_0 = 1$. The average emission number rate of electrons and positrons from such a black hole is [6]

$$\frac{dN}{dt} \approx 9.516 \times 10^{-4} M_{bh}^{-1}, \tag{2.1}$$

and the average emission power is

$$\frac{dE}{dt} \approx 1.589 \times 10^{-4} M_{bh}^{-2}. \tag{2.2}$$

The ratio of the power to the number rate gives the average energy per particle,

$$\langle E_e \rangle \equiv m_e \gamma_{av} = \frac{dE}{dN} \approx 0.167 M_{bh}^{-1} \approx 4.20\, T_{bh}. \tag{2.3}$$

This defines an average gamma factor, $\gamma_{av} \approx 4.20\, T_{bh}/m_e$, for the emitted electrons and positrons. The size of the black hole, $2M_{bh}$, is much less than the reduced Compton wavelength of the particles, $1/m_e$.

We can then write the average emission number rate in terms of γ_{av} as

$$\frac{dN}{dt} = \nu m_e \gamma_{av} \approx 5.70 \times 10^{-3} m_e \gamma_{av}, \tag{2.4}$$

thereby defining the dimensionless numerical constant

$$\nu \equiv \frac{dN/dt}{dE/dN} = \frac{(dN/dt)^2}{dE/dt} \approx 5.70 \times 10^{-3} \tag{2.5}$$

that we shall use later. Our assumption of $T_{bh} \gg m_e$ implies that $\gamma_{av} \gg 1$, and so almost all of the emitted electrons and positrons are ultrarelativistic. Because they are moving outward at very nearly the speed of light, they have an average radial separation in the black hole frame of $(dN/dt)^{-1}$, which is a factor of $1/\nu \approx 175$ times larger than their Lorentz-contracted reduced Compton wavelength $1/(m_e\gamma_{av})$. Thus the electrons and positrons are, on average, widely separated once they are emitted by the black hole.

Since the electrons and positrons have a highly inhomogeneous density distribution outside the black hole (going essentially as $1/r^2$ for $r \gg M_{bh}$) and a highly anisotropic momentum distribution (diverging essentially radially away from the black hole), it is inappropriate to use formulae such as $\langle n\sigma v \rangle$ for the bremsstrahlung production rate. Instead, we shall estimate the expected fraction F of the average charged particle energy $m_e\gamma$ that goes into bremsstrahlung photons from the electromagnetic scattering of two ultrarelativistic charged particles emitted by a black hole with $T_{bh} \gg m_e$. We will do this by first estimating the corresponding fraction f from the scattering of an electron or positron (the 'scattered particle') by another individual charged particle (the 'scattering particle,' also emitted by the black hole as part of its Hawking radiation), and then summing over all such potential scattering particles.

Note that f and F are expectation values in the quantum mechanical sense and are *not* the expected fraction of the energy of an individual interacting charged particle that goes into photons when one or more photons are radiated by that particle. The emitted photon will typically have an energy comparable to that of the ultrarelativistic charged particle emitting it (excluding the infrared divergence in the total number of photons of infinitesimally small energy) [7]. However, for the two-body bremsstrahlung around the black hole, we find that the probability of the emission of any such photon is of the order of $\alpha^3 \nu \sim 10^{-8}$. This is very tiny, so when this probability is multiplied by the fraction of the charged particle energy carried off by the photon, one gets expected fractions f and F that are $O(10^{-8})$ or less. The reason that we shall focus on the expected energy fractions f and F, rather than on the bremsstrahlung emission probabilities, is that f and F, being weighted by the photon energies, avoid the infrared divergences in the emission probabilities at infinitesimally small photon energies.

The bremsstrahlung process occurs over a formation length scale $\sim \gamma/m_e \sim \gamma^2 M_{bh}$ in the center of momentum frame of the interaction. This is much greater than the black hole radius $2M_{bh}$, so we shall approximate the black hole as a spatial point (timelike worldline) in flat spacetime that emits charged particles at the emission rate and power given above. For simplicity we shall assume that each charged particle is emitted with the same energy, corresponding to the gamma factor $\gamma_{av} \approx 4.20\,T_{bh}/m_e$. The expected fraction f of the energy of the charged particle lost as bremsstrahlung photons in Coulomb scattering off another charged particle will depend on the initial quantum states of the two particles. Since we have found that black hole emission leads to particles widely separated compared to their Lorentz-contracted reduced Compton wavelengths, we can use wavepackets in which the particles travel with constant velocities in straight lines from the point black hole worldline. Then, with γ having already been assumed, f just depends on the emission angle θ between the scattering particle and the scattered particle, and on the time separation δt between the emission of the scattering and the scattered particle.

We shall approximate f in the $T_{bh} \gg m_e$ black hole case by using the results of Sec. 4 of Bethe and Heitler [7] for the bremsstrahlung radiation probability as a function of impact parameter. Their calculation is for a fixed scattering particle (e.g., an atomic nucleus that is assumed not to recoil), but the results for f should be roughly the same (i.e., to within a factor of 2 or so) when the scattering and scattered particles have comparable mass (e.g., when both are relativistic electrons and/or positrons). It is convenient to express their results for the radiation probability as a function of impact parameter by giving f as a function of the orbital angular momentum ℓ where ℓ is the impact parameter multiplied by the linear momentum: The linear momentum is roughly $m_e\Gamma$ for $\Gamma \gg 1$, where Γ is the gamma factor of the scattered particle in the frame of the scattering particle. Then, when the atomic screening is dropped as not relevant for our case, the Bethe-Heitler Eqs. (39), (38), and (38A) respectively imply that the fraction of the initial energy of the electron emitted as bremsstrahlung photons is

$$
\begin{aligned}
f &\sim A\alpha^3[\ln(\Gamma/\ell) + B] \quad \text{for} \quad 1 \ll \ell \ll \Gamma; \\
f &\approx (2/\pi)\alpha^3\Gamma^2/\ell^2 \quad \text{for} \quad \Gamma \ll \ell \ll \Gamma^2; \\
f &\approx C\alpha^3\Gamma^4/\ell^3 \quad \text{for} \quad \Gamma^2 \ll \ell.
\end{aligned}
\tag{2.6}
$$

The numerical constants A, B, and C are presumably $O(1)$, although they do not appear to have been explicitly calculated in the literature. The normalization of the middle equation has been chosen to make the total cross section match the dominant contribution of Bethe-Heitler Eq. (16) in the ultrarelativistic limit. One can also deduce the middle equation (though not the other two) from the results of von Weizsäcker [8].

One can readily calculate that $C = \pi/4$ by the classical formula for the energy emitted by an accelerating charge e as the time integral of $(2/3)e^2a^2$, where a^2 is the square of the 4-acceleration. For accelerated motion in the Coulomb field of another charge e in the limit that the bending angle is very small, this gives [9]

$$
f \approx (\pi/4)\alpha^3(\Gamma^2 - 1/3)(\Gamma^2 + 1)/\ell^3.
\tag{2.7}
$$

We can then fit all three equations in (2.6) by the following single equation, after making a simple *ad hoc* choice for B and using $\ell + 1$ instead of ℓ in the logarithm to avoid a divergence at $\ell = 0$:

$$
f(\Gamma, \ell) \sim \frac{2\alpha^3}{\pi c}\ln\left[1 + c\left(\frac{\Gamma}{\ell + 1}\right)^2\right]\frac{\pi^2\Gamma^2}{\pi^2\Gamma^2 + 8\ell},
\tag{2.8}
$$

where the constant c is to be determined.

Now we can solve for c (and hence A) by matching the subleading term in the energy-averaged total cross section. From the Bethe-Heitler [7] Eq. (16) for the differential cross section for the emission of a photon of energy k by an electron of high initial energy $E_0 = \Gamma m_e \gg m_e$, one can readily calculate that the energy-averaged total cross section is

$$
\langle\sigma\rangle = \int \frac{k}{E_0}\frac{d\sigma}{dk}dk \approx \frac{4\alpha^3}{m_e^2}\left[\ln(2\Gamma) - \frac{1}{3}\right],
\tag{2.9}
$$

where all terms with inverse powers of Γ have been omitted. This must give the same answer as

$$\langle\sigma\rangle = \frac{\pi}{m_e^2(\Gamma^2-1)}\sum_\ell (2\ell+1)f(\Gamma,\ell) \approx \frac{4\alpha^3}{m_e^2}\left[\ln\left(\frac{\pi^2}{8}\Gamma\right)+\frac{1}{2}-\frac{1}{2}\ln c\right]. \tag{2.10}$$

Equating the right-hand sides of Eqs. (2.9) and (2.10) implies that $c = \pi^4 e^{5/3}/256$ and $A = 2^9/(\pi^5 e^{5/3})$ (where here e is the base of Napierian logarithms and not the magnitude e of the electron charge), in agreement with our earlier remark that A is $O(1)$. Therefore, we can write Eq. (2.8) with no undetermined parameters as

$$f(\Gamma,\ell) \sim \frac{2^9\alpha^3}{\pi^5 e^{5/3}}\ln\left[1+\frac{\pi^4 e^{5/3}}{256}\left(\frac{\Gamma}{\ell+1}\right)^2\right]\frac{\pi^2\Gamma^2}{\pi^2\Gamma^2+8\ell}. \tag{2.11}$$

Of course, the precise form of this approximation is rather *ad hoc*, such as its inclusion of the addition of 1 to ℓ in the logarithm to avoid a divergence at $\ell = 0$, and the rational function used as the final factor, but we would expect this formula to give the right order of magnitude for all ℓ when $\Gamma \gg 1$. It would be interesting to do a partial wave analysis of bremsstrahlung by a Coulomb potential to derive a more precise approximation in terms of the energy Γm_e of the incoming electron and its orbital angular momentum ℓ, especially for $\ell \ll \Gamma$ where the relative error of Eq. (2.11) is likely to be largest, though that regime contributes relatively little in our use of this formula below.

Ignoring factors of the order of unity and logarithms of possibly large numbers like $\Gamma/(\ell+1)$, the expected fraction of the available energy that goes into bremsstrahlung photons (essentially the probability that a photon is emitted with energy comparable to that of the scattered particle) is then $f \sim \alpha^3$ if the scattered particle gets within its reduced Compton wavelength $1/m_e$ of the scattering particle, and the fraction drops rather rapidly with the impact parameter if it is much greater.

III. CAUSALITY SUPPRESSION OF TWO-BODY BREMSSTRAHLUNG

The Bethe-Heitler bremsstrahlung results are for a charged particle coming in from infinity and scattering off a stationary Coulomb potential. (The Bethe-Heitler approach also applies the Born approximation of neglecting multiple scatterings off the potential, which should be adequate for our purposes [5].) However, the bremsstrahlung from the scattering of particles emitted by a black hole is produced by particles that do not come in from infinity, but rather are emitted by the black hole. In our approximation that the black hole is a point worldline at the spatial origin of flat spacetime, both the scattering particle and the scattered particle have worldlines that start at the location of the black hole, rather than coming in from infinity and passing by each other with some impact parameter. We are not certain how to handle this case precisely, but we expect the right order of magnitude for an upper estimate of the bremsstrahlung simply by replacing the impact parameter in the Bethe-Heitler results with the minimum distance, D, of the scattered particle from the scattering particle, in the frame of the scattering particle that replaces the static Coulomb field of the Bethe-Heitler formula.

Of course, the scattered particle will not detect any influence from the scattering particle until there is time for a causal signal to travel from the scattering particle to the scattered particle, after the scattering particle has been emitted by the black hole. (Before the causal signal arrives, the scattered particle would detect the charge of the scattering particle as part of the charge of the black hole, and we are not considering this here.) Therefore, D should be the minimum distance, in the frame of the scattering particle, to the scattered particle after the scattered particle can receive a causal signal from the scattering particle.

Let us now calculate this distance D. It will be a function of the gamma factor γ of the particles, the angle θ between the directions of the scattering particle and the scattered particle in the black hole frame, and the time delay δt between the emission of the scattering particle and the scattered particle in the black hole frame (which can be negative or positive). We need to join the scattering particle worldline, at spacetime 4-vector position $\mathbf{X}(\tau_1)$ and 4-velocity $\mathbf{u} = d\mathbf{X}/d\tau_1$, with the scattered particle worldline, at position $\mathbf{Y}(\tau_2)$ and 4-velocity $\mathbf{v} = d\mathbf{Y}/d\tau_2$, by a future-directed null line segment $\mathbf{N} = \mathbf{Y} - \mathbf{X}$ (representing the signal carrying the electric field from the scattering particle to the scattered particle). We then take the dot product of this null line segment with the scattering particle 4-velocity in order to get the distance $D = \mathbf{N} \cdot \mathbf{u}$ from the scattering particle to the scattered particle in the frame of the former. The result can be most simply expressed in terms of δt, γ, and the relative gamma factor Γ or energy per rest mass of one of the particles in the frame of the other,

$$\Gamma \equiv \mathbf{u}\cdot\mathbf{v} = \gamma^2 - (\gamma^2-1)\cos\theta. \tag{3.1}$$

Here we have assumed $\gamma_1 \approx \gamma_2 \approx \gamma$, which is true for the strongly-peaked ultrarelativistic Hawking spectrum.

If the scattered particle is emitted first, so $\delta t < 0$, then the minimum distance D is obtained by having the null line segment begin on the scattering particle immediately after it is emitted by the black hole, giving

$$D = \left[(\Gamma - 1)\gamma + \Gamma\sqrt{\gamma^2 - 1} \right] (-\delta t). \tag{3.2}$$

On the other hand, if the scattering particle is emitted first, so $\delta t > 0$, then the minimum distance D is obtained by having the null line segment end on the scattered particle immediately after it is emitted by the black hole, giving

$$D = \sqrt{\gamma^2 - 1}\, \delta t. \tag{3.3}$$

Therefore, if the minimum distance is to be less than some value $D_{\max}$, the relative difference in the emission times of the scattering and scattered particles must lie in the range

$$\Delta t = \delta t_{\max} - \delta t_{\min} = \frac{D_{\max}}{\sqrt{\gamma^2 - 1}} + \frac{D_{\max}}{(\Gamma - 1)\gamma + \Gamma\sqrt{\gamma^2 - 1}} \approx \frac{D_{\max}}{\gamma} \frac{2\Gamma}{2\Gamma - 1}, \tag{3.4}$$

where the approximation applies because $\gamma \gg 1$. Since $\Gamma \geq 1$ for all values of θ and $\Gamma \gg 1$ over most of the angular range, the factor $2\Gamma/(2\Gamma - 1)$ lies between 1 and 2 and is usually near 1. Hence, if we want the minimum distance D between the scattering particle and the scattered particle in the frame of the former (after there has been time for a signal to go from the former to the latter and after both have been emitted from the black hole) to be no greater than some $D_{\max}$, then the emission time in the black hole frame of the latter, relative to that of the former, must occur within a range that is roughly $\Delta t \approx D_{\max}/\gamma$.

In particular, for D to be not much greater than $1/m_e$, so that f is not much smaller than $O(\alpha^3)$, Δt must not be much greater than $1/(m_e\gamma_{av})$. This corresponds to only about $\nu \approx 1/175$ of the average time between the successive emissions of charged particles. Therefore, it would be rare for even one particle to be emitted soon enough to undergo a scattering that produces a bremsstrahlung photon of significant energy, even with probability $O(\alpha^3)$, from any putative scattering particle.

If one uses the approximation of Eq. (2.11) for f (though the result below is dominated by the part where $\Gamma \ll \ell \ll \Gamma^2$, where this formula is known to be good), assumes that the fraction of the scattered particle energy lost is roughly the same in the black hole frame (where F is defined) as in the scattering particle frame (which should be true except for the small fraction of cases in which both particles are emitted in nearly the same direction from the black hole), and replaces ℓ by $m_e\Gamma D$, then the total fraction of the scattered particle energy that is emitted into bremsstrahlung photons becomes

$$F \approx \int f\, dN \approx \frac{1}{m_e\gamma_{av}} \frac{dN}{dt} \sum_{\ell=0}^{\infty} \frac{1}{\Gamma} f(\Gamma, \ell) \approx \frac{2\alpha^3}{m_e\gamma_{av}} \frac{dN}{dt} = 2\alpha^3\nu \approx 4.43 \times 10^{-9}. \tag{3.5}$$

This is independent of the black hole mass, provided its temperature is much greater than the electron mass. (At lower temperatures the charged particle emission rate, and hence F, is exponentially suppressed to even smaller values by Boltzmann factors in the Hawking distribution [6].) As the black hole becomes hotter than the masses of other species of charged particles, there will be similar additional contributions from them. However, provided the number of species is not too large, one will always have $F \lesssim 10^{-7}$.

Because of the various approximations and *ad hoc* assumptions that have been made to derive our expectation value of the fraction F of the charged particle energy going into bremsstrahlung from two-particle scattering, we estimate that it has an uncertainty of at least a factor of 2. Furthermore, our approach may give an overestimate by an even greater factor, because the total momentum transfered by the Coulomb field of the scattering particle to the scattered particle after they are both emitted by the black hole must be less than the corresponding momentum transfer in the case of two particles coming in from infinity, to the same minimum distance, by a factor η that is classically always less, and sometimes much less, than one-half.

To estimate η classically, consider for example the case of $\delta t > 0$, so that the scattering particle is emitted first. This case dominates the estimate for F above. Then when the scattered particle comes out from the black hole and first receives a signal from the scattering particle, it is, in the frame of the scattering particle, already at a positive angle ϕ beyond where the position of closest approach to the scattering particle would have been, if both particles had had constant-velocity worldlines coming in from infinity.

It is convenient to express this angle ϕ in terms of the celerity p (the spatial distance per proper time, or spatial momentum per rest mass) of a particle with velocity $v \approx \sqrt{\gamma^2 - 1}/\gamma$ in the black hole frame,

$$p \equiv \gamma v = \sqrt{\gamma^2 - 1}, \tag{3.6}$$

the sine of half the angle between the two particles in the black hole frame,

$$s \equiv \sin \frac{\theta}{2}, \tag{3.7}$$

and the relative velocity (velocity of one particle in the frame of the other),

$$V \equiv \frac{\sqrt{\Gamma^2 - 1}}{\Gamma} = \frac{2ps\sqrt{p^2s^2 + 1}}{2p^2s^2 + 1}. \tag{3.8}$$

Then

$$\sin \phi = \frac{1}{V}\left[1 - \frac{1}{\mathbf{u} \cdot \mathbf{v}} \frac{\mathbf{N} \cdot \mathbf{v}}{\mathbf{N} \cdot \mathbf{u}}\right] = \sqrt{\frac{p^2s^2 + s^2}{p^2s^2 + 1}}. \tag{3.9}$$

If the two particles had come in from infinity with relative velocity V and with impact parameter b, and if one approximates their trajectories as straight lines with constant velocities, the magnitude of the 4-momentum transfer classically would be $\Delta P \approx 2e^2/(Vb)$, and b would be (approximately) the distance of nearest approach. Similarly, a classical calculation of the magnitude of the momentum transfer by two particles emitted from the black hole and thereafter traveling with constant velocity (with D the closest distance of the scattered particle from the scattering particle once a signal has traveled from the latter to the former), gives the magnitude of the 4-momentum transfer as

$$\Delta P \approx \frac{2e^2}{VD}\sqrt{\frac{1/2}{1 + \sin\phi} - \frac{1}{4}V^2} \approx \frac{e^2}{2\gamma^2 \delta t}\sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}}, \tag{3.10}$$

where the second approximation of (3.10) applies for $\Gamma = 2p^2s^2 + 1 \gg 1$ or $1 - \cos\theta \gg 1/p^2 \approx 1/\gamma^2$. This implies that the classical momentum transfer in the black hole case corresponds to that for two particles coming in from infinity with impact parameter $b = D/\eta$, where the reduction factor for the magnitude of the 4-momentum transfer is

$$\eta \equiv \frac{VD\Delta P}{2e^2} \approx \sqrt{\frac{1/2}{1 + \sin\phi} - \frac{1}{4}V^2} \approx \frac{1}{4\gamma}\sqrt{\frac{1 + \cos\theta}{1 - \cos\theta}}. \tag{3.11}$$

Replacing ℓ by $m_e\Gamma b = m_e\Gamma D/\eta$, rather than by $m_e\Gamma D$, we then get the following smaller estimate for the total fraction of the scattered particle energy that is emitted into bremsstrahlung photons:

$$F \approx \frac{\pi\alpha^3\nu}{4\gamma_{av}}. \tag{3.12}$$

Thus this classical estimate of F is suppressed by a factor of $\pi/(8\gamma_{av})$, i.e., by the inverse of the charged particle average gamma factor. This suggests that the average fractional energy loss to bremsstrahlung may be significantly lower than the already small estimate of 4×10^{-9} of Eq. (3.5). However, if we include the finite size of the black hole and nonradial motions of the emitted particles near the hole, the fractional energy loss is probably decreased by less than the gamma factor of (3.12), though it may still be significantly lower than the conservative upper bound given by Eq. (3.5).

Since our estimate for F given by Eq. (3.5), which should be taken as a rough upper limit on the average fraction of energy lost to two-particle bremsstrahlung by a charged particle emitted from a $T_{bh} \gg m_e$ black hole, is more than 8 orders of magnitude smaller than unity, it strongly indicates that the two-body bremsstrahlung radiated by the scattering of the charged particles emitted by a black hole is completely negligible and insufficient to form a photosphere by many orders of magnitude. This strong suppression comes primarily from our inclusion of the causality constraint which was omitted in the Heckler photosphere model. Applications to QCD are discussed in [5].

The causality suppression that we have calculated here may also be viewed as a partial justification of the approximation used in [10] of only including scattering of partons that approach each other. That paper concluded that TeV higher-dimensional black holes that might be created by the CERN LHC will not form chromospheres.

IV.   INNER BREMSSTRAHLUNG EMISSION

There are several other radiative processes that are also generally smaller than the direct photon emission but comparable to or larger than the two-particle bremsstrahlung analyzed above. Most important is inner bremsstrahlung

[11, 12, 13] from the charged particles that are emitted from the black hole. To an observer at infinity, each charged emitted particle appears to have its velocity change from rest at the position of the hole to an ultrarelativistic outward asymptotic velocity. The change in the electromagnetic field of the emitted particle from the Coulomb field corresponding to the initial velocity to the Coulomb field corresponding to the final velocity should appear to a distant observer as inner bremsstrahlung radiation. The analogous calculation of electromagnetic radiation from an individual relativistic charged particle falling into a neutral black hole was performed in Ref. [14].

In the black hole frame, it is most probable that the inner bremsstrahlung photon will be emitted in very nearly the same direction as that of the charged particle that emits it. If the photon carries a fraction f of the energy $m_e\gamma$ of the charged particle in the black hole frame and hence has wavelength $(fm_e\gamma)^{-1}$, it will become separated from the charged particle by one wavelength after traveling outward by a distance $\sim (fm_e\gamma)^{-1}/(1 - v_e) \sim \gamma/(fm_e)$ from the black hole. This exceeds the size of the hole, $\sim M_{bh} \sim 1/T_{bh} \sim 1/(m_e\gamma)$, by a factor of roughly γ^2/f. This distance, roughly $\gamma^2 M_{bh}/f$, is the effective formation length for the bremsstrahlung photon [15]. Since it is so much larger than the size of the black hole (for $\gamma \gg 1$), we may simply apply the standard flat spacetime analysis [11, 12, 13].

The inner bremsstrahlung process involves just one photon vertex, so the fraction of the particle's energy expected to be radiated (i.e., the probability that a charged particle of a given energy will emit a photon of comparable energy) is $O(\alpha)$. This is small, but not nearly so small as the $O(\alpha^3)$ bremsstrahlung analyzed above arising from the interaction of two charged particles which involves three photon vertices. The total power in the inner bremsstrahlung photons will be less than the power in the direct photon emission by a factor $O(\alpha)$. Since the direct photon emission has a power per frequency interval that goes as the fourth power of the frequency at low frequencies [16], whereas the inner bremsstrahlung photons have a white spectrum at frequencies well below the black hole temperature, the white inner bremsstrahlung photons will dominate at photon energies below $\sim \alpha^{1/4} T_{bh}$.

More precisely, the number flux of inner bremsstrahlung photons of energy ω radiated by particles of mass m and charge $\pm e$ that were emitted from the black hole with a spectrum $d^2N_\epsilon/dtdE$ is [11]

$$
\begin{aligned}
\frac{d^2N_{b\gamma}}{dtd\omega} &= \frac{2\alpha}{\pi\omega}\sum_\epsilon \int_m^\infty dE \frac{d^2N_\epsilon}{dtdE}\left[\frac{E^2 + 2\epsilon m(E-\omega) + (E-\omega)^2}{2(E+\epsilon m)\sqrt{E^2-m^2}}\ln\frac{E-\omega+\sqrt{(E-\omega)^2-m^2}}{m} - \frac{\sqrt{(E-\omega)^2-m^2}}{\sqrt{E^2-m^2}}\right] \\
&\approx \frac{2\alpha}{\pi\omega}\left[\ln(2\gamma_{av})-1\right]\frac{dN}{dt},
\end{aligned}
\tag{4.1}
$$

where $\epsilon = 2(j-\ell) = \pm 1$ for total angular momentum j and orbital angular momentum ℓ, and where the approximation of the last line of Eq. (4.1) applies for $\gamma_{av} \approx E/m \gg 1 + \omega/m$. Multiplying Eq. (4.1) by ω gives the corresponding spectrum for the power of the inner bremsstrahlung photons. The power spectrum from a charged particle of initial energy E is very nearly flat until the photon energy ω approaches its maximum value of $E - m$, where the spectrum rapidly but smoothly drops to zero. The total power radiated in inner bremsstrahlung photons by charged particles emitted from the black hole with power dE/dt is then

$$
\frac{dE_{b\gamma}}{dt} \approx \frac{2\alpha}{\pi}\left[\ln(2\gamma_{av})-1\right]\frac{dE}{dt}.
\tag{4.2}
$$

The mean gamma factor $\gamma_{av} \approx 4.20 T_{bh}/m$ is given by Eq. (2.3) for ultrarelativistic charged spin-half particles emitted by a black hole of temperature $T_{bh} \gg m$ (or $M_{bh} \ll 10^{17}(m_e/m)$ g).

We may then use Eqs. (2.1) and (2.2) to compare the inner bremsstrahlung photon power $dE_{b\gamma}/dt$ with the direct photon power given in [16] and [17], which is

$$
\frac{dE_{d\gamma}}{dt} \approx 0.3364 \times 10^{-4} M_{bh}^{-2}.
\tag{4.3}
$$

For example, for a black hole with $T_{bh} = 50$ GeV, the gamma factors are $411\,000$, 1986, and 118 respectively for electrons, muons, and taus, and so the sum of the three logarithmic square-bracket factors of Eq. (4.2) is 24.37. For the ratio of the inner bremsstrahlung photon power to the direct photon power, one must multiply this logarithmic factor by

$$
\frac{2\alpha}{\pi}\frac{dE/dt}{dE_{d\gamma}/dt} \approx (0.004646)(4.724) \approx 0.02195,
\tag{4.4}
$$

where $dE/dt \approx 1.589 \times 10^{-4} M_{bh}^{-2}$ from Eq. (2.2) represents the power in each species of ultrarelativistic spin-half charged particle and antiparticle, and $dE_{d\gamma}/dt \approx 0.3364 \times 10^{-4} M_{bh}^{-2}$ from Eq. (4.3) represents the power in photons directly emitted by the Hawking process. For a black hole with $T_{bh} = 50$ GeV, one then finds that the inner bremsstrahlung photons radiated by the electrons, muons, taus, and their antiparticles give 53% as much power as the directly emitted photons.

Thus for such a hot black hole, the sum of the logarithm factors (24.37), multiplied by the factor 4.734 for the greater power in spin-half particles and antiparticles than in direct photons (a factor of 2.362 from the lower centrifugal barrier for a black hole to emit particles of lower spin [6, 16, 17], multiplied by a factor of 2 for the inclusion of distinct antiparticles for the spin-half particles), nearly compensates for the small factor of $(2/\pi)\alpha \approx 0.004646$ from the single photon vertex of this inner bremsstrahlung process. When one includes the emission of other charged particles, such as pions and/or quarks, the total power in inner bremsstrahlung photons could possibly exceed the total power in photons directly emitted by the Hawking process for a sufficiently hot black hole.

If one considers a cooler black hole, say with temperature $T_{bh} \approx 21$ MeV and mass $M_{bh} \approx 5.0 \times 10^{14}$ g, whose lifetime equals the present age of the Universe [5, 16, 18], then only electrons and positrons are ultrarelativistic, with $M_{bh}m_e \approx 0.0010$. Muons and antimuons are partially relativistic, with $M_{bh}m_\mu \approx 0.20$, and taus have $M_{bh}m_\tau \approx 3.34$ or $m_\tau/T_{bh} = 8\pi M_{bh}m_\tau \approx 84$ and so are hardly emitted at all. In this case the electrons and positrons have $\gamma_{av} \approx 173.7$, while the muons and antimuons have $\gamma_{av} \approx 1.305$. The power in the ultrarelativistic electrons and positrons is given by Eq. (2.2). Reference [6] similarly gives the power in muons and antimuons when $M_{bh}m_\mu \approx 0.20$ as

$$\frac{dE}{dt} \approx 0.491 \times 10^{-4} M_{bh}^{-2}. \tag{4.5}$$

The power in muons and antimuons is only about 30% as large as for electrons and positrons because of the Boltzmann suppression in the Hawking distribution due to the greater mass of the muon. For $M_{bh} \approx 5.0 \times 10^{14}$ g, the ratio of inner bremsstrahlung to direct photon power is then

$$\frac{dE_{b\gamma}/dt}{dE_{d\gamma}/dt} \approx 0.108, \tag{4.6}$$

of which nearly 99% comes from the inner bremsstrahlung photons generated by the electrons and positrons, and about 1.3% comes from the muons and antimuons.

At low frequencies, the spectrum for the power per frequency interval from direct photons in the Hawking radiation is very blue, going as the fourth power of the frequency [16], whereas the inner bremsstrahlung spectrum is independent of the frequency (white), up to the cutoff at the energy of the emitting charged particles. Therefore, at sufficiently low frequency, the inner bremsstrahlung photons dominate over the direct photons. Using the results of Refs. [16] and [19], one can see that for $M_{bh} \approx 5.0 \times 10^{14}$ g, the inner bremsstrahlung photon spectrum dominates for $M_{bh}\omega < 0.107$, that is for photon energies $\omega < 57$ MeV, whereas the peak of the direct photon spectrum occurs at $M_{bh}\omega \approx 0.24$ and $\omega \approx 130$ MeV.

Equation (4.2) implies that the power spectrum of inner bremsstrahlung photons from a black hole of mass $M_{bh} \approx 5.0 \times 10^{14}$ g is

$$\frac{d^2 E_{b\gamma}}{dt d\omega} \approx 1.73 \times 10^{19}\,\mathrm{s}^{-1}. \tag{4.7}$$

The bremsstrahlung photons are cut off above an energy which is roughly the average energy of the electrons and positrons, about $4.20 T_{bh} \approx 90$ MeV from Eq. (2.3). If one integrates over the frequencies of photons in the visual range, say between 400 and 750 nm, one finds that the power in visible photons (almost entirely inner bremsstrahlung radiation) is about 4.0 W. The total power in all frequencies is about 2.56 gigawatts (about 6.4×10^8 times greater than that in the visible range) and peaks at around 130 MeV, where the directly emitted Hawking photons dominate. However, these direct photons have a spectrum of the form [16]

$$\frac{d^2 E_{d\gamma}}{dt d\omega} = \frac{8}{3\pi^2} M^3 \omega^4 \tag{4.8}$$

at low ω, giving a power in the visible range of only about 2.4×10^{-29} W. This is about 6.0×10^{-30} times that of the inner bremsstrahlung photons in the visible range and is less than 10^{-38} of the total photon power.

Since a star of visual magnitude $m_V = 6$, which is barely visible to the naked human eye, gives a visible photon energy flux of about 10^{-8} lux [20], where a lux is a lumen per square meter and a lumen at the visible wavelength 555 nm is 0.001 47 W [21], a barely visible light source emits a flux of visible photons of about $1.5 \times 10^{-11}\,\mathrm{Wm}^{-2}$. Therefore, for a 5×10^{14} g black hole, generating 4 W of power in the visible spectrum by inner bremsstrahlung, to be just visible to the human eye, it must be within a distance of about 150 km. However, at that distance, there will be a flux of about $0.01\,\mathrm{Wm}^{-2}$ of high-energy gamma rays. This would lead to the recommended maximum yearly dosage in a human, of the order of 1 rem [22], which deposits an energy of $0.01\,\mathrm{Jkg}^{-1}$ (see p. 120 of Ref. [21]) in a time of the order of 4 minutes. In roughly half a day, one would receive about 200 rem, which "will cause vomiting in 50% of those exposed after about 3 hours," and in roughly a full day, one would receive about 450 rem, which

is "the radiation dose that gives a 50% probability of death ... for healthy people without medical treatment" [23]. Therefore, one would not want to stay unprotected very long close enough to a 5×10^{14} g black hole to be able to see it without a telescope.

The proposed 100-meter Overwhelmingly Large Telescope (OWL), whose concept is now being studied by the European Southern Observatory, "will be able to reach magnitude 38 in 10 hours exposure time. This is a factor five thousand billion ... fainter than the faintest star visible to the naked eye" [24]. In principle OWL should see the inner bremsstrahlung visual photons from a 5×10^{14} g black hole at a distance of about 4×10^8 km or so, or about 3 AU, roughly the distance from Earth to the Asteroid Belt. At this distance the gamma ray flux would be less than $2 \times 10^{-15} \, \mathrm{Wm^{-2}}$, which would be quite safe for humans even without shielding.

V. OTHER BREMSSTRAHLUNG PROCESSES

Besides the inner bremsstrahlung that comes from the change as viewed from infinity in the velocity of a charged particle as it is emitted out of a black hole, another effect is the bremsstrahlung emitted when a charged particle coming out of the black hole scatters off the electric field of the hole itself. As the black hole emits electrons and positrons stochastically, it will have an rms charge $O(1)$ in Planck units. (In Planck units with $4\pi\epsilon_0 = 1$, the positron charge is $e = \alpha^{1/2}$.) For example, Ref. [6] showed that for a black hole of $T_{bh} \ll m_e$, the rms charge is approximately $1/\sqrt{8\pi} \approx 0.1995 \approx 2.335e$, whereas for $T_{bh} \gg m_e$, as is applicable for our calculations, the rms charge is $\approx 0.5247 \approx 6.143e$. The average square of the photon vertex with the black hole is then of order unity. However, to calculate the fraction of the power in these bremsstrahlung photons, there will be one photon vertex on the scattering particle involving the Coulomb field of the black hole and one vertex involving the emitted bremsstrahlung photon. Thus the fraction of the power will be $O(\alpha^2)$. This is one power of α smaller than the inner bremsstrahlung but is still one power of α larger than the two-particle bremsstrahlung analyzed in Secs. II and III.

Another bremsstrahlung process is the scattering by a charged particle of the inner bremsstrahlung photon emitted by another charged particle. This process gives outgoing photons of expected power fraction $O(\alpha^3)$ and so would be comparable to the two-particle bremsstrahlung analyzed above, but two powers of α smaller than the inner bremsstrahlung itself. This process could be described classically by noting that the retarded field of the scattering particle does not have a purely Coulomb form (corresponding to constant velocity of the scattering particle) at the retarded time when it is just coming out from the black hole. As it is being emitted, the charged particle is effectively accelerated from being at rest at the black hole position (approximating the black hole as a point in flat spacetime) to having its asymptotic post-emission velocity. Since this scattered bremsstrahlung (the photons emitted by the inner bremsstrahlung and then scattered by another charged particle) is $O(\alpha^2)$ smaller than the inner bremsstrahlung directly emitted by the scattering particle, it is already known to be small. Thus we can ignore it and take the bremsstrahlung from the scattering particle to be that found above by assuming the particle to have achieved its asymptotic velocity and hence to have a retarded field which is purely Coulombic, with electric field e/d^2 where the distance d is measured in the frame of the scattering particle (D being the minimum value of d).

There are also other radiative effects, discussed in Ref. [6], of smaller order, such as the vacuum polarization by the black hole with its fluctuating charge, and self-energy corrections to the propagation of the charged particles. The latter effect comes from the fact that the charged particles will be surrounded by clouds of virtual photons and so will not propagate in the black hole spacetime exactly the same way as the point Dirac particles numerically analyzed in Ref. [6].

VI. DISCUSSION

All of these effects analyzed in this paper are generally small in comparison with the direct emission of particles and photons from the black hole (except in comparison with photons well below their peak frequency), and none of these effects will lead to photospheres. In particular we have confirmed that the two-particle bremsstrahlung is far too weak to lead to a QED photosphere.

An intuitive way of seeing our result, that the fraction of energy going into two-body bremsstrahlung photons is not enhanced above the α^3 value that one would get from simply counting photon vertices, even for large gamma factors, is as follows: If one regards each electron as being a three-dimensional blob of radius $1/m_e$ (its Compton radius) in its own rest frame, then as the electrons are emitted by a black hole with gamma factors $\gamma \sim T_{bh}/m_e \gg 1$ and their Coulomb fields propagate at finite speed, the electrons are Lorentz-contracted in the radial direction to thickness $\sim 1/(m_e\gamma)$ and their shapes distorted to resemble eggshells in the black hole frame. This thickness is less than the average radial separation between the electrons being emitted by the black hole by a factor of about $1/\nu \approx 175$. (This factor is a number of order unity in the sense that it contains no powers of any physical quantities like α or m_e but just comes from numerical solutions [6] of the Dirac equation around a black hole.) Roughly speaking, bremsstrahlung

occurs with probability $\sim \alpha^3$ if two electron blobs overlap, but since the overlap has low probability $\sim \nu \approx 5.70 \times 10^{-3}$ here, the probability is even lower than α^3. In particular, there is no enhancement when γ gets large, since the factor of $1/\gamma$ in the Lorentz contraction precisely cancels the factor of γ in the emission rate.

Another way of expressing the smallness of the two-body bremsstrahlung is the following: Because the bremsstrahlung formation length, given in Sec. IV as $\sim \gamma^2 M_{bh}/f$, is so much larger than the size of the black hole, what is most relevant for the spectrum of the bremsstrahlung photons is the change in the momentum of each charged particle over this distance. Since each charged particle effectively starts at rest at the black hole from the perspective of a viewer at infinity, the change in its spatial momentum over this distance is very nearly its asymptotic momentum. Therefore, the bremsstrahlung spectrum (including the inner bremsstrahlung and the various corrections to it from the scattering from the fluctuating charge of the black hole and from other charged particles that are emitted) can be calculated purely from the asymptotic momentum distribution of the charged particles emitted by the black hole. A very good approximation for this momentum distribution is given simply by using the Hawking emission formula with numerical solutions for the particle wavefunctions propagating in the field of the black hole, such as was done for charged spin-half particles in Ref. [6]. In this approach, the bremsstrahlung can be regarded purely as inner bremsstrahlung with fractional energies of $O(\alpha)$. The Coulomb scattering of the charged scattered particles from other charged scattering particles in the outgoing flux then gives $O(\alpha^2)$ corrections to the asymptotic momentum distribution and hence to the bremsstrahlung spectrum, leading to an absolute effect of $O(\alpha^3)$ for the fraction of energy going into bremsstrahlung photons. From this viewpoint, the scattering between different charged particles emitted by the black hole gives only a tiny $O(\alpha^2)$ correction to the inner bremsstrahlung and so is effectively negligible.

Our results here confirm what was written in Ref. [6], "Since the average time between the emission of successive leptons will turn out to be greater than $10^3 M$, it should be a very good approximation to ignore the interactions between different leptons emitted." In particular, the interactions between charged particles emitted by a black hole are almost completely negligible and cannot form a QED photosphere as Heckler [3, 4] has claimed. Similar implications for QCD chromospheres are discussed in [5, 10].

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